(Defn.) Cauchy - Riemann Partial differential equation:- If $w = f(z) = u + iv$, where $u = u(x, y)$, $v = v(x, y)$, $u_x = v_y$, $u_y = -v_x$, are known as Cauchy Riemann Partial differential equations.

M.U. / M.Sc. If The Necessary Condition for $f(z)$ to be analytic

Q.No→ Establish Cauchy - Riemann equations.

or, Q.No→ State and Prove Cauchy - Riemann equation Partial differential equation.

Statement:- If a function $f(z) = u(x, y) + iv(x, y)$ is differentiable at any Point $z = x + iy$, then the Partial derivatives u_x, v_x, u_y, v_y should exist and Satisfy the equations.

$$u_x = v_y,\ u_y = -v_x.$$

Proof:- Since $f(z) = u(x, y) + iv(x, y)$ is ~~differ~~ differentiable at ~~Point~~ $z = x + iy$.

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \text{ exists}$$

Since, $z = x + iy$

$\therefore \Delta z = \Delta x + i\Delta y$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) + iv(x, y)}{\Delta x + i\Delta y} \quad \text{---(1)}$$

exists

If $\Delta z \to 0$ along the real line, then $\Delta y = 0$ the above limit will be

$$= \lim_{\Delta x \to 0} \frac{\{u(x+\Delta x, y) - u(x,y)\} + i\{v(x+\Delta x, y) - v(x,y)\}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left\{ \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \frac{i\{v(x+\Delta x, y) - v(x,y)\}}{\Delta x} \right\}$$

$$= u_x + i v_x \quad \text{---} \quad (2)$$

Again, if Δz approches zero through imaginary axis.

i.e. when $\Delta x = 0$ the above limit, becomes

$$\lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + i v(x, y+\Delta y) - u(x,y) - i v(x,y)}{i \Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i \Delta y} + \frac{i v(x, y+\Delta y) - v(x,y)}{i \Delta y}$$

$$= \frac{u_y}{i} + v_y = -i u_y + v_y \quad \text{---} \quad (3)$$

Since, the function is differentiable

$x + iy$, we have

$$u_x + i v_x = -i u_y + v_y$$

$$\therefore u_x = v_y \,;\; u_y = -v_x \quad \text{---} \quad (4)$$

The eqn. given in (4) are known as the Cauchy Riemann Partial differential equation.

Other method:-

Ans→ **Statement**:- If $f(z)$ is differentiable at any point $z = x + iy$, then the real and imaginary parts u, v of $f(z)$ are also differentiable at (x, y) and, moreover,

$$u_x = v_y, \quad u_y = -v_x$$

Proof:- Since $f(z)$ is differentiable at z, we have a relation of the form

$$f(z+h) - f(z) = hf'(z) + h\epsilon \quad \text{--- (1)}$$

where, ϵ depends upon h & $\to 0$ as $h \to 0$.

We write,

$$h = p + iq, \quad f'(z) = \alpha + i\beta, \quad \epsilon = \epsilon_1 + i\epsilon_2$$

Then, we can re-written (1) as

$$[u(x+p, y+q) - u(x,y)] + i[v(x+p, y+q) - v(x,y)]$$
$$= (p+iq)(\alpha + i\beta) + (\epsilon_1 + i\epsilon_2)(p+iq) \quad \text{--- (2)}$$

Equating real and imaginary parts in (2), we obtain

$$u(x+p, y+q) - u(x,y) = p\alpha - q\beta + p\epsilon_1 - q\epsilon_2 \quad \text{--- (3)}$$
$$v(x+p, y+q) - v(x,y) = p\beta + q\alpha + q\epsilon_2 + q\epsilon_1 \quad \text{--- (4)}$$

The relations (3) & (4) show that the functions $u(x,y)$ and $v(x,y)$ are differentiable at (x,y) and also,

$$u_x = \alpha, \quad u_y = -\beta \quad \text{--- (5)}$$
$$v_x = \beta, \quad v_y = \alpha \quad \text{--- (6)}$$

From (5) & (6), we obtain

$$u_x = v_y, \quad u_y = -v_x \quad \text{--- (7)}$$

thus connecting the partial derivatives of u & v, the two partial differential eq^n (7) are known as Cauchy-Riemann partial differential equations.

Method of Constructing a regular function (Milne-Thomson's Method:-

Let $f(z) = u(x,y) + iv(x,y)$ ~~and $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2}$~~

Since, $z = x + iy$, $\bar{z} = x - iy$

$\therefore x = \frac{1}{2}(z+\bar{z})$, $y = \frac{1}{2i}(z-\bar{z})$

$$\therefore f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) \quad \text{—(1)}$$

Since (1) is a formal identity in two independent variables it will be satisfied for every value of z & $\bar{z}$

Putting $z = \bar{z}$, we have

$f(z) = u(z,0) + iv(z,0)$

$\therefore f'(z) = u_x + iv_x$

$= u_x - iu_y$ [by using Cauchy-Riemann equations]

In this case, we have $x = z$, $y = 0$

Let $u_x = \phi_1(x,y)$, $u_y = \phi_2(x,y)$

$\therefore u_x = \phi_1(z,0)$; $u_y = \phi_2(z,0)$

$\therefore f'(z) = \phi_1(z,0) - i\phi_2(z,0)$

Integrating, we get

$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c$

where c = arbitrary Constt.

Similarly, if $v(x,y)$ be given, then

$f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)]\,dz + c'$

where, $v_y = \psi_1(x,y)$ & $v_x = \psi_2(x,y)$.

Ex. Q.N. → Find the analytic function of which the real part is
$$e^{-x}\{(x^2-y^2)\cos y + 2xy \sin y\}$$

Soln. Here,
$u = e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\}$, where $f(z) = u+iv$

$$\therefore \frac{\partial u}{\partial x} = -e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\} + e^{-x}\{2x\cos y + 2y\sin y\} = \phi_1(x, y)$$

$$\therefore \phi_1(z, 0) = -e^{-z}.z^2 + e^{-z}.2z = (2z - z^2)e^{-z}$$

$$\frac{\partial u}{\partial y} = e^{-x}\{-2y\cos y - (x^2-y^2)\sin y + 2x.(1.\sin y + y\cos y)\} = \phi_2(x, y)$$

$$\therefore \phi_2(z, 0) = e^{-z} \times 0 = 0.$$

$$\therefore f(z) = \int (2z - z^2)e^{-z}\,dz + C$$
$$= -(2z - z^2)e^{-z} + \int e^{-z}(2-2z)\,dz + C.$$
$$= (z^2 - 2z)e^{-z} - (2-2z)e^{-z} - 2\int e^{-z}\,dz + C$$
$$= (z^2 - 2z - 2 + 2z + 2)e^{-z} + C$$
$$= z^2 e^{-z} + C.$$

Ex. Q.N. - If $u = e^{x}(x\cos y - y\sin y)$, find the analytic function $u + iv$.

Soln. Here, we have

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$
$$= e^{-x}(x\sin y + \sin y + y\cos y)\,dx + e^{x}(x\cos y - y\sin y + \cos y)\,dy$$

$\therefore v = \int_y e^x(x\sin y + \sin y + y\cos y)\,dx + \int$ (those terms which do not contain x) $dy + C$

$= (x\sin y + \sin y + y\cos y)\cdot e^x - e^x\sin y + C$

$= e^x(x\sin y + y\cos y) + C$, where C = Constt.

$\therefore f(z) = u + iv = e^x[x\cos y - y\sin y + ix\sin y - iy\cos y] + Ci$

$= e^x(x + iy)(\cos y + i\sin y) + Ci$

$= e^{x+iy}\cdot(x + iy) + Ci$

$= ze^z + Ci$

QN.→ Prove Polar form of Cauchy-Riemann equation.

Proof:- If $f(x, y)$ is ~~cartesian~~ cartesion and (r, θ) be the Polar Co-ordinates of a Pointe in the organ Plane then,

$x = r\cos\theta,\ y = r\sin\theta$

$\therefore x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$

and $\tan\theta = \dfrac{y}{x}$ $\therefore \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

$\therefore \dfrac{\partial r}{\partial x} = \dfrac{2x}{2r} = \dfrac{x}{r} = \cos\theta$

and, $\dfrac{\partial\theta}{\partial x} = \dfrac{1}{1+\dfrac{y^2}{x^2}} \times \left(-\dfrac{y}{x^2}\right) = -\dfrac{y}{x^2+y^2} = -\dfrac{r\sin\theta}{r^2} = -\dfrac{\sin\theta}{r}$

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{2y}{2r} = \dfrac{y}{r} = \sin\theta$

$\dfrac{\partial\theta}{\partial y} = \dfrac{1}{1+\dfrac{y^2}{x^2}}\cdot\dfrac{1}{x} = \dfrac{x^2}{x^2+y^2}\cdot\dfrac{1}{x} = \dfrac{x}{x^2+y^2} = \dfrac{r\cos\theta}{r^2} = \dfrac{\cos\theta}{r}$

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial\theta}\cdot\dfrac{\partial\theta}{\partial x}$

$= \dfrac{\partial u}{\partial r}\cos\theta - \dfrac{\partial u}{\partial\theta}\cdot\dfrac{\sin\theta}{r}$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

$$= \frac{\partial u}{\partial r} \sin\theta + \frac{\partial u}{\partial \theta} \cdot \frac{\cos\theta}{r}$$

Similarly, it we can be shown that,

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} = \frac{\partial v}{\partial r} \cos\theta - \frac{\partial v}{\partial \theta} \cdot \frac{\sin\theta}{r}$$

and $$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta} \cdot \frac{\partial \theta}{\partial y} = \frac{\partial v}{\partial r} \sin\theta + \frac{\cos\theta}{r} \cdot \frac{\partial v}{\partial \theta}$$

From Cauch- Riemann - Partial Diff- equation we know that,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\therefore \frac{\partial u}{\partial r} \cdot \cos\theta - \frac{\sin\theta}{r} \cdot \frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} \sin\theta \cdot \frac{\cos\theta}{r} \frac{\partial v}{\partial \theta} \quad \text{--- (1)}$$

and $$\frac{\partial u}{\partial r} \sin\theta + \frac{\cos\theta}{r} \cdot \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \cos\theta + \frac{\sin\theta}{r} \cdot \frac{\partial v}{\partial \theta} \quad \text{--- (2)}$$

From (1) x $\cos\theta$, + (2) x $\sin\theta$, we have

$$\frac{\partial u}{\partial r}(\cos^2\theta + \sin^2\theta) = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta}(\cos^2\theta + \sin^2\theta)$$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta} \quad \text{--- (3)}$$

From (1) x $-\sin\theta$ + (2) x $\cos\theta$, we have

$$\frac{1}{r} \cdot \frac{\partial u}{\partial \theta}(\cos^2\theta + \sin^2\theta) = -\frac{\partial v}{\partial r}(\cos^2\theta + \sin^2\theta)$$

$$\therefore \frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

$$\text{i.e. } \frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r} \quad \text{--- (4)}$$

Equations (3) & (4) are the Cauchy-Riemann eqn in Polar form.

We can now obtain the derivative of w in Polar form, we have

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$$= \frac{\partial w}{\partial r} \cdot \cos\theta - \left(\frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta}\right) \cdot \frac{\sin\theta}{r}$$

$$\cancel{= \left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r}\right) \cos\theta}$$

$$= \frac{\partial w}{\partial r} \cos\theta - \left(-r \frac{\partial v}{\partial r} - i \cdot r \frac{\partial u}{\partial r}\right) \frac{\sin\theta}{r}$$

[from (3) & (4)]

$$= \frac{\partial w}{\partial r} \cos\theta - i \left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r}\right) \sin\theta$$

$$= \frac{\partial w}{\partial r} \cos\theta - i \frac{\partial w}{\partial r} \sin\theta$$

$$\frac{dw}{dz} = (\cos\theta - i\sin\theta) \frac{\partial w}{\partial r} = e^{-i\theta} \frac{\partial w}{\partial r}$$

Ex. Q No → If $w = f(z) = u + iv$ and $u - v = e^x(\cos y - \sin y)$ find w in terms of z.

Soln. Here, $u - v = e^x(\cos y - \sin y)$.

$$\therefore \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x(\cos y - \sin y) \quad \text{--- (1)}$$

and $$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = -e^x(\sin y + \cos y)$$

or, $$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = -e^x(\sin y + \cos y)$$

[using Cauchy Riemann equations]

or, $$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = e^x(\sin y + \cos y) \quad \text{--- (2)}$$

Solving ① & ②, we get

$\frac{\partial u}{\partial x} = e^{x}\cos y = \phi_1(x, y)$, $\frac{\partial v}{\partial x} = e^{x}\sin y = \phi_2(x, y)$

and So, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \phi_1(x, y) + i\phi_2(x, y)$.

$\therefore f(z) = \int [\phi_1(z, 0) + i\phi_2(z, 0)]\, dz$

$= \int e^{z} dz + c = e^{z} + c$.

QNo→ If $u = (x-1)^3 - 3xy^2 + 3y^2$, determine v so that $u + iv$ is a regular function of $x + iy$.

Solⁿ:- It given that,

$u = (x-1)^3 - 3xy^2 + 3y^2$

$\therefore \frac{\partial u}{\partial x} = 3(x-1)^2 - 3y^2$

and $\frac{\partial u}{\partial y} = -6xy + 6y$

From Cauchy-Riemann eqⁿ, we know that,

$u_y = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -6xy + 6y$

$\therefore \frac{\partial v}{\partial x} = 6xy - 6y$.

Therefore, by integrating w.r.to x, we get

$v = 3x^2 y - 6xy + f(y)$ ——— ①

$\therefore \frac{\partial v}{\partial y} = 3x^2 - 6x + f'(y)$

By the Cauchy Riemann eqⁿ, we know that

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\therefore 3(x-1)^2 - 3y^2 = 3x^2 - 6x + f'(y)$$

$$\therefore f'(y) = 3 - 3y^2$$

$$\cancel{\therefore f(y) = 3y - y^3}$$

Integrating, we get

$$f(y) = 3y - y^3 + c.$$

Substituting this value of $f(y)$ in ①, we have

$$v = 3x^2 y - 6xy + 3y - y^3 + c.$$